THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 50, No. 5, May-June, 1971
Printed in U.S.A.

Optimum Equalization and the Effect of Timing and Carrier Phase on Synchronous Data Systems

By E. Y. HO

(Manuscript received December 16, 1970)

The minimum mean-square error (M.M.S.E.) at the receiver output generally depends upon the sampling instant and demodulating carrier phase for synchronous data systems. In this study, it is shown that for certain single-sideband data systems with no excess bandwidth (e.g., class IV and class V partial-response systems), the M.M.S.E. is completely independent of the sampling instant and demodulating carrier phase if the receiver contains an infinitely long transversal filter equalizer. Practically speaking, computer calculations indicate that for a class IV system operating in the presence of typical received signal-to-noise ratios, a 19-tap equalizer is sufficient to make the M.M.S.E. relatively insensitive to the sampling instant and demodulating carrier phase. Thus, for such data systems, a significant reduction in the receiver complexity and possibly in the start-up time may be obtained, because no time is spent acquiring timing and carrier phase.

The optimum infinite-length equalizer for synchronous data systems with a fixed channel is also calculated for two different conditions. The conditions are: (i) the minimization of the output noise plus mean-square intersymbol interference and (ii) the minimization of the output noise subject to the constraint that the equalizer forces the intersymbol interference to zero. Explicit expressions for the optimum equalizer and the M.M.S.E. are obtained. Satisfying condition (i) results in the lower value of M.M.S.E.; however, the M.M.S.E.s for these two criteria are almost equivalent for either large signal-to-noise ratios or small slope of the amplitude-frequency characteristics of the channel.

I. INTRODUCTION

In synchronous data systems, the transmission rates are frequently limited by the intersymbol interference which is caused by the ampli-

tude and phase distortion in the transmission channel. In order to reduce the effect of the intersymbol interference, it is necessary to equalize the channel before the data can be transmitted.

Several automatic equalization schemes using transversal filters have been devised for such data systems.[1–5] Chang[6] has investigated the effect of the sampling instant and carrier phase on the minimum mean-square intersymbol interference for a noiseless system with a finite-length transversal equalizer. In principle, we can make the mean-square intersymbol interference arbitrarily small by using an infinitely long transversal filter, provided that the tap-gain settings can be made arbitrarily accurate. However, the equalizer which forces the intersymbol interference to zero may not be the most desirable one when noise is present.

We have found, in this study, the optimum infinite-length mean-square equalizer for such synchronous data systems with a fixed channel. Two different cases are considered corresponding to the following optimality criteria: (*i*) the minimization of the output noise plus mean-square intersymbol interference, and (*ii*) the minimization of the output noise power subject to the constraint that the equalizer forces the intersymbol interference to zero.

Explicit expressions for the optimum equalizer and the M.M.S.E. are obtained. We also have found that for certain types of data systems (S.S.B. class IV or class V partial-response systems) the M.M.S.E. does not depend upon sampling instant and demodulating carrier phase. Thus, for such data systems, it may be possible to reduce significantly the receiver complexity and the start-up time.

A computer program has been written for a class IV system which is equipped with a finite-length mean-square equalizer. The number of taps needed to achieve near optimum performance in practical situations can then be determined. Throughout, additive Gaussian noise and independence of information digits are assumed.

II. GENERAL CONSIDERATIONS

The optimality criteria will be formulated in this section. A simplified block diagram of a general digital data system is shown in Fig. 1. We assume that every T seconds, an impulse of amplitude a_n ($a_n = \{2M + 1, \cdots 1, -1, \cdots -(2M + 1)\}$), is transmitted to the input of the system. The a_n are assumed to be identically distributed independent random variables.

In the absence of channel noise, for a sequence of input impulses

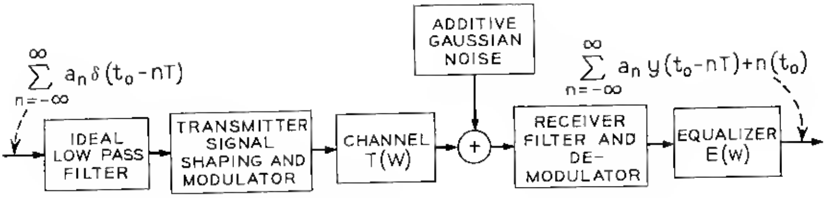

Fig. 1—Digital data system, simplified block diagram.

$$\sum_{n=-\infty}^{\infty} a_n \delta(t - nT), \tag{1}$$

the corresponding sequence at the receiver equalizer output is

$$\sum_{n=-\infty}^{\infty} a_n y(t - nT, \theta), \tag{2}$$

where $y(t, \theta)$ is the system impulse response with demodulating carrier phase θ.*

With noise, the output at the sampling instant t_0 with a demodulating carrier phase θ_0 is

$$V = a_0 y(t_0, \theta_0) + \sum_{n \neq 0} a_n y(t_0 - nT, \theta_0) + n(t_0) \tag{3a}$$

or

$$V = a_0 y_0(\theta_0) + \sum_{n \neq 0} a_n y_n(\theta_0) + n_0, \tag{3b}$$

where the terms $y_n(\theta_0)$, $n \neq 0$, represent intersymbol interference and $y_0(\theta_0)$ is the output value at the main sample point.

One useful measure of the performance of such a data system is mean-square error (M.S.E.). In this study, we define the normalized M.S.E. at the sampling instant t_0 with a demodulating carrier phase θ_0 to be

$$\begin{aligned}[\text{M.S.E.}]_{t_0,\theta_0} &= E\{[V - a_0 y_0(\theta_0)]^2\}/E\{a_0^2 y_0^2(\theta_0)\} \\ &= [E\{n_0^2\} + E[a_j^2] \sum_{n \neq 0} y_n^2(\theta_0)]/E\{a_0^2 y_0^2(\theta_0)\},\end{aligned} \tag{4}$$

where $E[X]$ means expectation of random variable X.

For binary systems, $E[a_j^2]$ is equal to 1. The variance of the noise at

* $\theta = 0°$ corresponds to the phase of the frequency component of the received spectrum at the carrier frequency.

the equalizer output (see the Appendix) is

$$\sigma_0^2 = E(n_0^2) = \frac{\sigma_i^2 T}{\pi} \int_0^{\pi/T} \psi \text{ eq } (\omega) \cdot | E(\omega) |^2 \, d\omega, \tag{5}$$

where $\sigma_i^2 T$ is the power spectral density of the input white noise, ψ eq (ω) is the square of the equivalent baseband receiver filter characteristic and $E(\omega)$ is the transfer function of the equalizer.

Now we wish to design two optimum equalizers for two different conditions. In the first the $[\text{M.S.E.}]_{t_0, \theta_0}$ is minimized subject to the constraint that $y_0(\theta_0)$ is a constant,

$$C_1 = y_0(\theta_0) = \text{Re} \frac{T}{\pi} \int_0^{\pi/T} Y \text{ eq } (\omega, \theta_0) E(\omega) e^{i\omega t_0} \, d\omega, \tag{6}$$

where Y eq (ω, θ_0) (see the Appendix) is the equivalent baseband system transfer function for sampling instant t_0 and demodulating carrier phase θ_0. In the second case the optimum equalizer is found by minimizing the variance of the output noise subject to the constraint equations, (6) and

$$0 = y_n(\theta_0) = \text{Re} \frac{T}{\pi} \int_0^{\pi/T} Y \text{ eq } (\omega, \theta_0) E(\omega) e^{i\omega(t_0 + nT)} \, d\omega, \quad n \neq 0. \tag{7}$$

III. MINIMIZATION OF NOISE PLUS INTERSYMBOL INTERFERENCE

3.1 *A General Binary Data System*

The details of the minimization of the $[\text{M.S.E.}]_{t_0, \theta_0}$ given by equation (4) subject to the constraint equation (6) are given in the Appendix.

For a binary data system, the optimum equalizer, $E_0(\omega)$, for sampling instant t_0 and demodulating carrier phase θ_0 is

$$[E_0(\omega)]_{t_0, \theta_0} = \frac{\{Y \text{ eq } (\omega, \theta_0) e^{i\omega t_0}\}^*}{\{\sigma_i^2 \psi \text{ eq } (\omega) + | Y \text{ eq } (\omega, \theta_0) |^2\}} \cdot \frac{C_1}{\dfrac{T}{\pi} \displaystyle\int_0^{\pi/T} \frac{| Y \text{ eq } (\omega, \theta_0) |^2}{\sigma_i^2 \psi \text{ eq } (\omega) + | Y \text{ eq } (\omega, \theta_0) |^2} \, d\omega} \tag{8}$$

where $\{X\}^*$ means complex conjugate of X. It follows that the M.M.S.E. for the corresponding sampling instant and demodulating carrier phase is

$$[\text{M.M.S.E.}]_{t_0, \theta_0} = \frac{1}{\dfrac{T}{\pi} \displaystyle\int_0^{\pi/T} \frac{| Y \text{ eq } (\omega, \theta_0) |^2}{\sigma_i^2 \psi \text{ eq } (\omega) + | Y \text{ eq } (\omega, \theta_0) |^2} \, d\omega} - 1. \tag{9}$$

$|Y\text{ eq }(\omega, \theta_0)|^2$ is, in general, a function of t_0 and θ_0. It follows that $[\text{M.M.S.E.}]_{t_0,\theta_0}$ depends generally upon t_0 and θ_0. Hence, there exists an optimum sampling instant $\bar{t}_0$ and demodulating carrier phase $\bar{\theta}_0$, such that

$$[\text{M.M.S.E.}]_{\bar{t}_0,\bar{\theta}_0} = \min_{\text{all } t_0,\theta_0} [\text{M.M.S.E.}]_{t_0,\theta_0}. \tag{10}$$

However, there exist cases where $|Y\text{ eq }(\omega, \theta_0)|^2$ is not a function of t_0 and θ_0; for example, the $|Y\text{ eq }(\omega, \theta_0)|^2$ of a single-sideband system with no excess bandwidth is independent of the sampling instant, t_0, and the demodulating carrier phase, θ_0, (see the Appendix.) The SSB class IV partial-response system represents another example. Therefore, if an infinite equalizer is available, no loss in performance occurs when arbitrary t_0 and θ_0 are used.

3.2 *SSB Class IV Partial-Response System*

In this section we will show that the M.M.S.E. for a class IV partial-response system is independent of sampling instant and demodulating carrier phase. The M.M.S.E. will be computed for certain typical telephone channels. Such a data system has been fully described in Reference 7.

The transfer functions of the transmitter and receiver filters are

$$S(\omega_c - \omega) = R(\omega_c - \omega)$$
$$= \begin{cases} \sqrt{2T \sin |\omega| T}\, e^{-j(\omega/|\omega|)\pi/4} & 0 < |\omega| \leqq \dfrac{\pi}{T}, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The M.S.E. for the partial-response system is defined to be

$$[\text{M.S.E.}]_{C,t_0,\theta_0} = \frac{[\sigma_0^2 + (y_1(\theta_0) + y_{-1}(\theta_0))^2 + \sum_{n\neq 1,-1} y_n^2(\theta_0)]}{[(y_1(\theta_0) - y_{-1}(\theta_0))/2]^2}. \tag{12}$$

The constraint equations are

$$C = y_1(\theta_0) = \text{Re}\,\frac{2T}{\pi}\int_0^{\pi/T} -j(\sin\omega T)T(\omega_c - \omega)e^{-j\theta_0}E(\omega)e^{j\omega(t_0+T)}\,d\omega \tag{13}$$

and

$$C - 2 = y_{-1}(\theta_0)$$
$$= \text{Re}\,\frac{2T}{\pi}\int_0^{\pi/T} -j(\sin\omega T)T(\omega_c - \omega)e^{-j\theta_0}E(\omega)e^{-j\omega(T-t_0)}\,d\omega \tag{14}$$

where Re $[X]$ means real part of X, $T(\omega)$ is the transfer function of the channel, and C is a constant.

For a given constant C, sampling instant t_0, and demodulating carrier phase θ_0, the optimum equalizer is

$$[E_0(\omega)]_{t_0,\theta_0,C} = \begin{cases} \dfrac{\left[\begin{array}{r} \dfrac{C(A_1 - A_2) + 2A_2}{A_1^2 - A_2^2}\, [-j2T(\sin T\,|\,\omega\,|)\cdot T(\omega_c - \omega) \\ \cdot e^{-j\theta_0}\cdot e^{j\omega(t_0+T)}]^* \\ + \\ \dfrac{C(A_1 - A_2) - 2A_1}{A_1^2 - A_2^2}\cdot [-j2T(\sin T\,|\,\omega\,|)\cdot T(\omega_c - \omega) \\ \cdot e^{-j\theta_0} e^{j\omega(t_0-T)}]^* \end{array}\right]}{\sigma_i^2 2T \sin T\,|\,\omega\,| + 4T^2 \sin^2 T\,|\,\omega\,|\cdot|\,T(\omega_c - \omega)\,|^2} & 0 \leqq |\,\omega\,| \leqq \dfrac{\pi}{T}, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where $[X]^*$ means complex conjugate of X, and, A_1 and A_2 are given by the equation (41a) and (41b) respectively.

It follows that, the corresponding M.M.S.E. is

$$\begin{aligned}[\text{M.M.S.E.}]_{C,t_0,\theta_0} &= \frac{1}{\pi}\int_0^{\pi/T} [\sigma_i^2 2T \sin \omega T + 4T^2 \sin^2 \omega T\cdot|\,T(\omega_c - \omega)\,|^2] \\ &\quad \cdot|\,[E_0(\omega)]_{C,t_0,\theta_0}\,|^2\, d\omega + 2C^2 - 4C. \end{aligned} \tag{16}$$

Since $|\,[E_0(\omega)]_{C,t_0,\theta_0}\,|^2$ is independent of t_0 and θ_0, the M.M.S.E. does not depend upon t_0 and θ_0. Equation (16) can be further minimized over all possible values of C. The optimum solution, $C = 1$, results in the minimum of the M.S.E. for the class IV partial-response system equipped with a mean-square equalizer.

$$\begin{aligned}[\text{M.M.S.E.}]_{C=1} &= \frac{2}{A_1 - A_2} - 2 \\ &= \min_{\text{all } C}\, [\text{M.M.S.E.}]_C \end{aligned} \tag{17}$$

We may thus conclude that for the SSB class IV partial-response system, the M.M.S.E. does not depend upon t_0 and θ_0 since the constants

A_i do not depend on t_0 and θ_0. We may therefore arbitrarily choose the sampling instant and the demodulating carrier phase with no loss of optimality as long as the equalizer transversal filter length is infinite.

As an example, we assume that the equivalent baseband channel characteristics are linear in amplitude-frequency response and quadratic in delay-frequency response as shown in Fig. 2. The delay at the Nyquist frequency of π/T rad/s is taken to be $\beta_m T$ seconds. The transfer function of the channel is

$$T(\omega_c - \omega) = \left(1 - \alpha \frac{|\omega|}{\pi/T}\right) e^{-j(\beta_m T^3 \omega^3 / 3\pi^2)} \qquad 0 \leqq |\omega| \leqq \frac{\pi}{T}. \tag{18}$$

The M.M.S.E.s computed by equation (17) for various α, $\beta_m T$, and σ_i^2 are given in Table I. For these calculations the transmitted signal power is fixed at $4/\pi$ watts.

3.3 *Computer Results for Class IV Partial-Response System with a Finite Length Equalizer*

A particular case has been calculated by the computer for a class IV partial-response system with a finite length equalizer. The following assumptions are made:

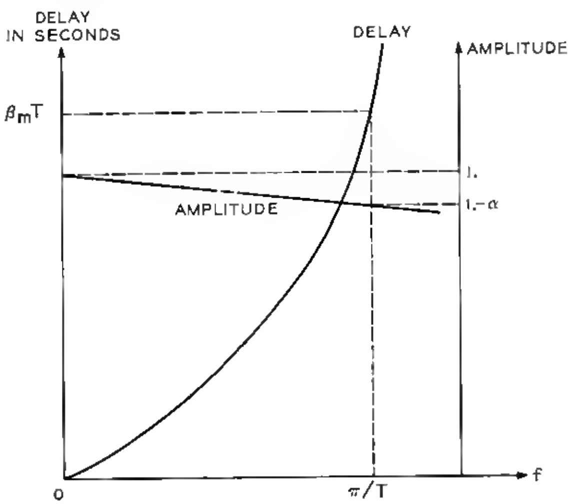

Fig. 2—Equivalent baseband channel characteristics.

TABLE I—M.M.S.E. COMPUTED BY EQUATION (17)

σ_i^2 (10^{-2} watts/Hz)	M.M.S.E. (10^{-2}) $\alpha = 0.1$, $\beta_m T = 1$, $T = 1$	M.M.S.E. (10^{-2}) $\alpha = 0.9$, $\beta_m T = 1$, $T = 1$
4	5.621	22.57
2	2.818	12.14
0.4	0.564	2.74
0.2	0.282	1.41

(*i*) The transfer functions of the channel is

$$T(\omega_c - \omega) = \left(1 - 0.1 \frac{|\omega|}{\pi/T}\right) e^{-j(\omega^3 \beta_m T^3/3\pi^2)} \qquad 0 \leqq |\omega| \leqq \pi/T. \quad (19)$$

(*ii*) The signal-to-noise ratio at the receiver input is assumed to be 21 dB.

(*iii*) The delay at the Nyquist frequency is taken to be 1 second and the baud is assumed to be 1 symbol/second.

Forty distinct combinations of sampling instants (0, 0.2, 0.4, 0.6, 0.8) and demodulating carrier phase (90°, 60°, 30°, 15°, 0°, −15°, −30°, −60°) have been tried with a 19-tap mean-square equalizer. The results and the minimum of the mean-square error are shown in Figures 3 through 10. It can be seen that the M.S.E. for most combinations is near the minimum achieved by the infinite equalizer. Practically speaking, in this example the system performance is acceptable (with error-rate upper bounded[8] by 10^{-8}) with a 19-tap mean-square equalizer for all 40 distinct combinations.

IV. MINIMIZATION OF NOISE SUBJECT TO THE CONSTRAINT THAT THE EQUALIZER FORCES THE INTERSYMBOL INTERFERENCE TO ZERO

4.1 *A General Binary Data System*

The minimization of the output noise power [see equation (5)],

$$\sigma_0^2 = \frac{\sigma_i^2 T}{\pi} \int_0^{\pi/T} \psi \text{ eq } (\omega) \cdot |E(\omega)|^2 \, d\omega,$$

subject to the constraint equations (6) and (7) can be solved through a straightforward application of the method of Lagrangian multipliers.

The expression of the optimum equalizer for sampling instant t_0 and demodulating carrier phase θ_0 is found to be

$$[E_0(\omega)]_{t_0, \theta_0} = \frac{C_1 R_d(\omega)}{Y \text{ eq } (\omega, \theta_0)} \qquad (20)$$

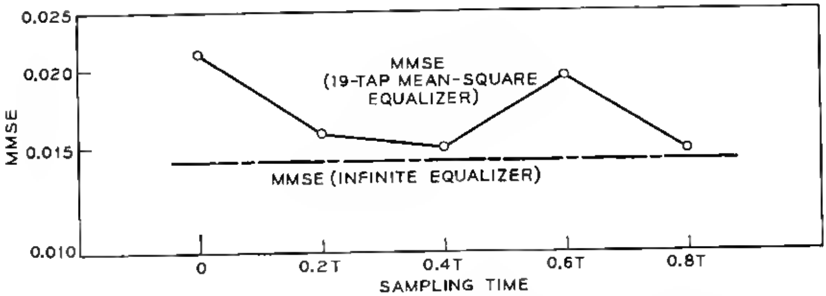

Fig. 3—M.M.S.E. versus sampling time; SSB class IV partial-response system; baseband equivalent channel transfer function, $\{1 - 0.1\ [|\omega|/(\pi/T)]\}$ $\exp - j(\omega^3\beta_m\ T^3/3\pi^2)$; demodulating carrier phase, $\theta = -60°$, $(S/N)_{\text{input}} =$ 21 dB.

where $R_d(\omega)$ is the desired received baseband equivalent signal spectrum.

It follows that the minimum output noise power is

$$[\min \sigma_0^2]_{t_0,\theta_0} = \frac{\sigma_i^2 T}{\pi} \int_0^{\pi/T} \psi \text{ eq } (\omega) C_1^2 \left| \frac{R_d(\omega)}{Y \text{ eq } (\omega, \theta_0)} \right|^2 d\omega. \qquad (21)$$

In general, Y eq (ω, θ_0) is a function of the sampling instant, t_0, and the demodulating carrier phase, θ_0; therefore the minimum output noise power depends upon t_0 and θ_0.

4.2 *Class IV Partial-Response System*

It can be seen from equation (21) that the M.M.S.E. generally depends upon t_0 and θ_0. However, for the SSB class IV partial-

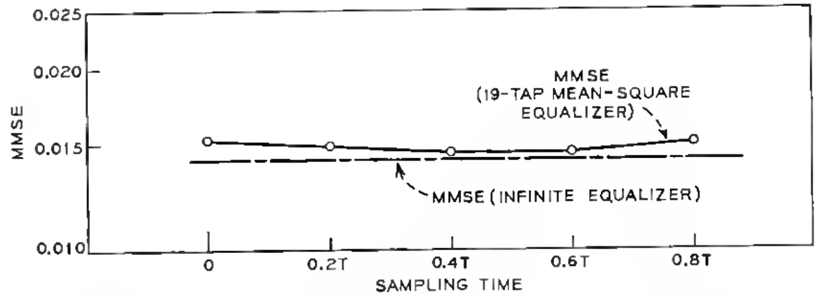

Fig. 4—M.M.S.E. versus sampling time; SSB class IV partial-response system; baseband equivalent channel transfer function, $\{1 - 0.1\ [|\omega|/(\pi/T)]\}$ $\exp - j(\omega^3\beta_m\ T^3/3\pi^2)$; demodulating carrier phase, $\theta = -30°$, $(S/N)_{\text{input}} =$ 21 dB.

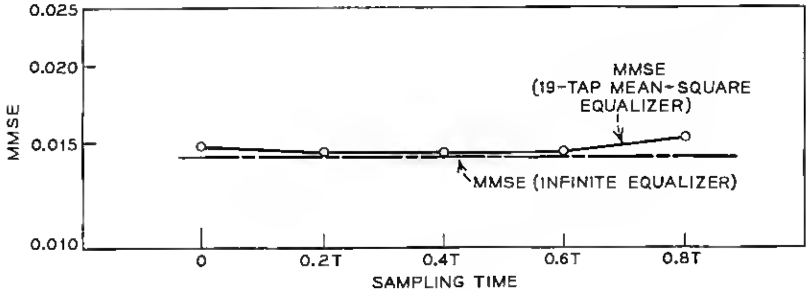

Fig. 5—M.M.S.E. versus sampling time; SSB class IV partial-response system; baseband equivalent channel transfer function, $\{1 - 0.1 [|\omega|/(\pi/T)]\} \exp - j(\omega^3\beta_m T^3/3\pi^2)$; demodulating carrier phase, $\theta = -15°$, $(S/N)_{\text{input}} = 21$ dB.

response system,

$$\left| \frac{R_d(\omega)}{Y \text{ eq } (\omega, \theta_0)} \right| = \left| \frac{1}{T(\omega_c - \omega)} \right|^2. \tag{22}$$

Therefore, the minimum output noise power is independent of t_0 and θ_0.

Table II shows the values of minimum output noise power computed by equations (21) and (22) for various α, $\beta_m T$, and σ_i^2 under the same assumptions made in Section III. For these calculations the transmitted signal power is fixed at $4/\pi$ watts.

Table III gives the difference in M.M.S.E. computed by equations (17) and (21).

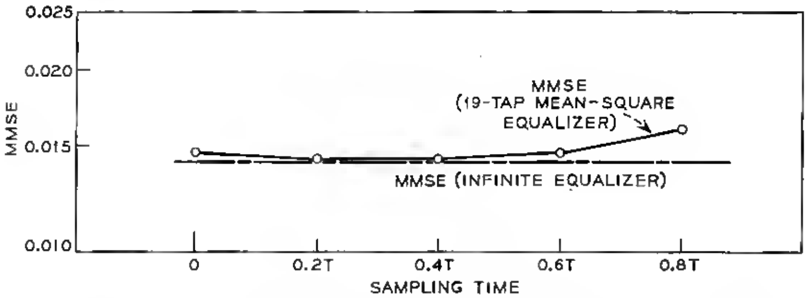

Fig. 6—M.M.S.E. versus sampling time; SSB class IV partial-response system; baseband equivalent channel transfer function, $\{1 - 0.1 [|\omega|/(\pi/T)]\} \exp - j(\omega^3\beta_m T^3/3\pi^2)$; demodulating carrier phase, $\theta = 0°$, $(S/N)_{\text{input}} = 21$ dB.

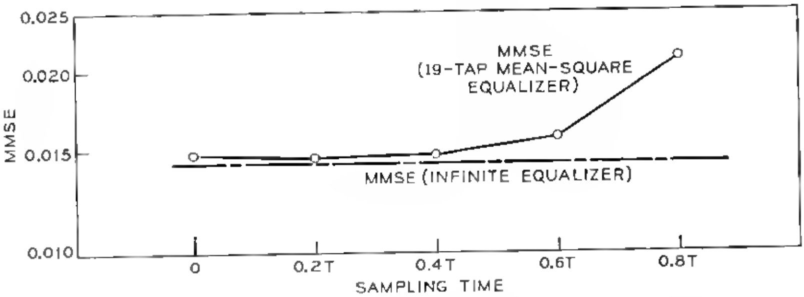

Fig. 7—M.M.S.E. versus sampling time; SSB class IV partial-response system; baseband equivalent channel transfer function, $\{1 - 0.1 [|\omega|/(\pi/T)]\}$ $\exp - j(\omega^3\beta_m T^3/3\pi^2)$; demodulating carrier phase, $\theta = 15°$. $(S/N)_{\text{input}} =$ 21 dB.

The results show that the M.M.S.E.s computed by equations (17) and (21) are almost the same if either the signal-to-noise ratio is large or the slope of the amplitude-frequency characteristic of the channel is small (e.g., in this case the slope is 0.1). Notice that either decreasing the signal-to-noise ratio or increasing the slope of the amplitude-frequency characteristic of the channel increases the disparity of the M.M.S.E.s obtained by (17) and (21). As an example, if

$$\sigma_i^2 = 0.02 \text{ watts/Hz}$$

and

$$T(\omega_c - \omega) = \left(1 - 0.9 \frac{|\omega|}{\pi}\right) e^{-j(\omega^3/3\pi^2)}, \tag{23}$$

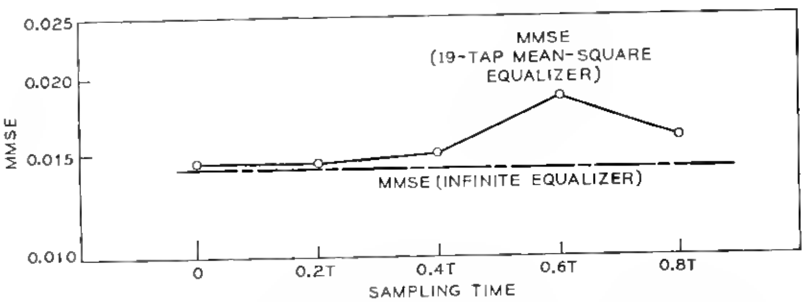

Fig. 8—M.M.S.E. versus sampling time; SSB class IV partial-response system; baseband equivalent channel transfer function, $\{1 - 0.1 [|\omega|/(\pi/T)]\}$ $\exp - j(\omega^3\beta_m T^3/3\pi^2)$; demodulating carrier phase, $\theta = 30°$, $(S/N)_{\text{input}} =$ 21 dB.

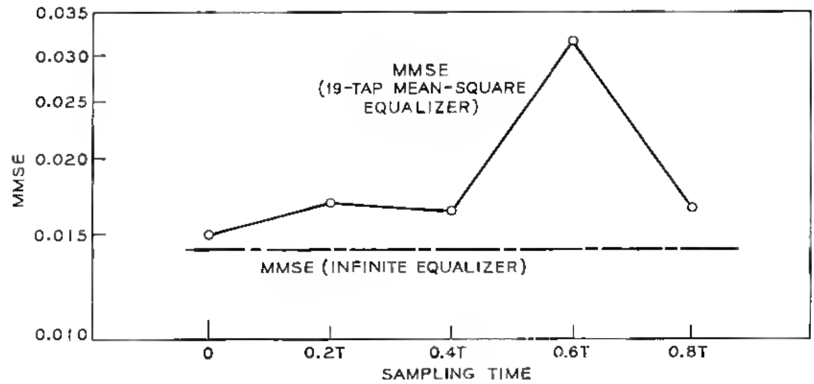

Fig. 9—M.M.S.E. versus sampling time; SSB class IV partial-response system; baseband equivalent channel transfer function, $\{1 - 0.1\ [|\omega|/(\pi/T)]\}$ $\exp - j(\omega^3\beta_m\ T^3/3\pi^2)$; demodulating carrier phase, $\theta = 60°$, $(S/N)_{\text{input}} =$ 21 dB.

then the M.M.S.E.s obtained by equations (17) and (21) are 0.1214 and 0.1483 respectively. It can be seen that the M.M.S.E. is 16 percent less if the equation minimizing the mean-square intersymbol interference plus noise is used.

V. SUMMARY AND CONCLUSION

The optimum equalizer for a synchronous data system with a fixed channel is derived in this study. Two different optimality criteria are assumed: (*i*) the minimization of the output noise plus mean-square

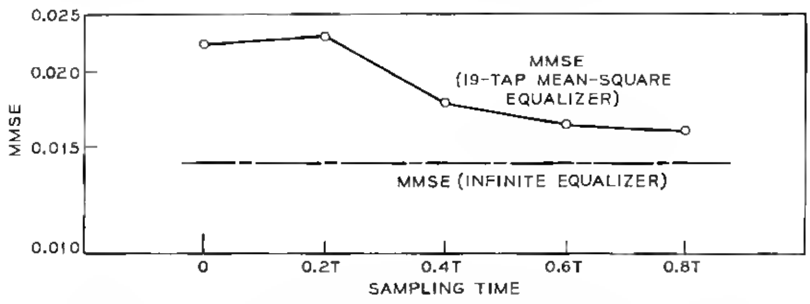

Fig. 10—M.M.S.E. versus sampling time; SSB class IV partial-response system; baseband equivalent channel transfer function, $\{1 - 0.1\ [|\omega|/(\pi/T)]\}$ $\exp - j(\omega^3\beta_m\ T^3/3\pi^2)$; demodulating carrier phase, $\theta = 90°$, $(S/N)_{\text{input}} =$ 21 dB.

TABLE II—MINIMUM OUTPUT NOISE POWER COMPUTED BY EQUATION (21)

σ_i^2 (10^{-2} watts/Hz)	σ_0^2 (10^{-2}) $\alpha = 0.1, \beta_m T = 1, T = 1$	σ_0^2 (10^{-2}) $\alpha = 0.9, \beta_m T = 1, T = 1$
4	5.652	29.66
2	2.826	14.83
1	1.413	7.42
0.4	0.565	2.97
0.2	0.282	1.49

intersymbol interference and *(ii)* the minimization of the output noise subject to the constraint that the equalizer forces the intersymbol interference to zero.

Explicit expressions for the optimum equalizer and the corresponding M.M.S.E. are obtained. It is known that the M.M.S.E. at the equalizer output generally depends upon the sampling instant and the demodulating carrier phase. However, we have shown in this study that there exist cases where the M.M.S.E. is independent of the sampling instant and the demodulating carrier phase. The SSB class IV partial-response system represents a good example. Thus for such data systems, we may use arbitrary timing and carrier phase, thereby significantly reducing the receiver complexity and possibly the start-up time as well. The results calculated by the computer for an SSB class IV partial-response system equipped with a 19-tap mean-square equalizer show that the system error-rate for all 40 distinct combinations of sampling instants and carrier phases is less than 10^{-8}. The system is operated over a channel with linearly distorted amplitude-frequency characteristic and parabolically distorted delay-frequency characteristic (see Section III) which is worse than a worst-case C-2 line. The signal-to-noise ratio at the receiver input is assumed to be 21 dB. The results also show that with either small slope of the

TABLE III—DIFFERENCE IN M.M.S.E.

σ_i^2 (10^{-2} watts/Hz)	σ_0^2 − M.M.S.E. (10^{-2}) $\alpha = 0.1, \beta_m T = 1, T = 1$	σ_0^2 − M.M.S.E. (10^{-2}) $\alpha = 0.9, \beta_m T = 1, T = 1$
4	0.031	7.08
2	0.08	2.69
1	0.002	0.97
0.4	≈0	0.23
0.2	≈0	0.08

amplitude-frequency characteristic of the channel or large signal-to-noise ratio, the M.M.S.E.s obtained by the two different criteria considered in this study are almost the same. For example, with white-noise spectral density 0.02 watts/Hz (S/N at the receiver input is 18 dB) and the Fourier transform of the channel

$$\left(1 - 0.1\frac{|\omega|}{\pi}\right)e^{-j(\omega^3/3\pi^2)},$$

the M.M.S.E.s obtained by criteria (*i*) and (*ii*) are 0.02818 and 0.02826 respectively. However, either increasing the slope or decreasing the signal-to-noise ratio increases the disparity of the M.M.S.E.s obtained by criteria (*i*) and (*ii*). Under these situations, criterion (*i*) is much preferred. For example, with the same white-noise spectral density as before and the Fourier transform of the channel

$$\left(1 - 0.9\frac{|\omega|}{\pi}\right)e^{-j(\omega^3/3\pi^2)},$$

the M.M.S.E.s obtained by criteria (*i*) and (*ii*) are 0.121 and 0.148 respectively.

VI. ACKNOWLEDGMENTS

The author wishes to thank J. E. Mazo, J. Salz, and D. A. Spaulding for their helpful discussions.

APPENDIX

Minimization of Noise Plus Intersymbol Interference (Binary and Partial-Response Systems)

The details of the minimization procedure of noise-plus-intersymbol interference for the binary and partial-response systems will be given in this Appendix.

The block diagram of a general digital data system is shown in Fig. 1. The characteristics of the ideal low-pass filter and the equalizer are assumed to be

$$F_1(\omega) = \begin{cases} 1 & 0 \leqq |\omega| \leqq \omega_m, \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

and

$$E(\omega) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega nT}.$$

With input white Gaussian noise having one-side power spectral density $T\sigma_i^2$ watts/Hz, the variance of the noise at the receiver equalizer output is,

$$\sigma_0^2 = E(n_0^2)$$
$$= \frac{T\sigma_i^2}{2\pi} \int_{-\omega m}^{\omega m} \{| R(\omega - \omega_c) |^2 + | R(\omega_c - \omega) |^2\}^* \cdot | E(\omega) |^2 \, d\omega$$
$$= \frac{T\sigma_i^2}{2\pi} \int_{-\omega m}^{\omega m} \psi(\omega) \, | E(\omega) |^2 \, d\omega$$
$$= \frac{\sigma_i^2 T}{2\pi} \left\{ \sum_{K=-N+1}^{N-1} \int_{(\pi/T)(2K-1)}^{(\pi/T)(2K+1)} \psi(\omega) \, | E(\omega) |^2 \, d\omega \right.$$
$$\left. + \int_{-\omega m}^{-(\pi/T)(+2N+1)} \psi(\omega) \, | E(\omega) |^2 \, d\omega + \int_{(\pi/T)(2N-1)}^{\omega m} \psi(\omega) \, | E(\omega) |^2 \, d\omega \right\}$$
$$= \frac{T\sigma_i^2}{\pi} \int_0^{\pi/T} \psi \text{ eq } (\omega) \cdot | E(\omega) |^2 \, d\omega, \tag{25}$$

where

$$\psi \text{ eq } (\omega) = \psi(\omega) + \psi\left(\omega + \frac{2\pi}{T}\right) + \cdots + \psi\left(\omega + \frac{2N\pi}{T}\right), \tag{26}$$

$$\frac{(2N - 1)\pi}{T} \leqq \omega_m \leqq \frac{(2N + 1)\pi}{T}.$$

Similarly,

$$y(t, \theta) = \frac{T}{2\pi} \int_{-\omega m}^{\omega m} [S(\omega - \omega_c)T(\omega - \omega_c)R(\omega - \omega_c)e^{i\theta}$$
$$+ S(\omega_c + \omega)T(\omega_c + \omega)R(\omega_c + \omega)e^{-i\theta}]E(\omega)e^{i\omega t} \, d\omega$$
$$= \frac{T}{2\pi} \int_{-\omega m}^{\omega m} Y(\omega, \theta)E(\omega)e^{i\omega t} \, d\omega$$
$$= \frac{T}{2\pi} \left[\int_{-\omega m}^{-(\pi/T)(2N+1)} Y(\omega, \theta)E(\omega)e^{i\omega t} \, d\omega \right.$$
$$+ \sum_{K=-N+1}^{N-1} \int_{(\pi/T)(2K-1)}^{(\pi/T)(2K+1)} Y(\omega, \theta)E(\omega)e^{i\omega t} \, d\omega$$
$$\left. + \int_{\pi/T(2N-1)}^{\omega m} Y(\omega, \theta)E(\omega)e^{i\omega t} \, d\omega \right]$$
$$= \text{Re} \frac{T}{\pi} \int_0^{\pi/T} Y \text{ eq } (\omega, \theta)E(\omega)e^{i\omega t} \, d\omega, \tag{27}$$

* $R(\omega)$ is the receiver filter transfer function.

where

$$Y \text{ eq } (\omega,\, \theta) = Y(\omega,\, \theta) + Y\left(\omega + \frac{2\pi}{T},\, \theta\right) + \cdots + Y\left(\omega + \frac{2N\pi}{T},\, \theta\right). \tag{28}$$

By Parseval's theorem we can write

$$\sum_n y_n^2(\theta) = \frac{T}{\pi} \int_0^{\pi/T} |\, Y \text{ eq } (\omega,\, \theta)\,|^2 \cdot |\, E(\omega)\,|^2\, d\omega. \tag{29}$$

Therefore the normalized M.M.S.E. given by equation (4) can be rewritten as

$$[\text{M.S.E.}]_{t_0,\theta_0} = \frac{\left[\dfrac{T\sigma_i^2}{\pi} \displaystyle\int_0^{\pi/T} \psi \text{ eq } (\omega) \cdot |\, E(\omega)\,|^2\, d\omega + B\right]}{y_0^2(\theta_0)}, \tag{30a}$$

where

$$B = \frac{T}{\pi} \int_0^{\pi/T} |\, Y \text{ eq } (\omega,\, \theta_0)\,|^2 \cdot |\, E(\omega)\,|^2\, d\omega - y_0^2(\theta_0). \tag{30b}$$

Since $y_0(\theta_0)$ is fixed, minimizing $[\text{M.S.E.}]_{t_0,\theta_0}$ subject to the constraint equation (6) is equivalent to minimizing the following function,

$$V = \frac{T}{\pi} \int_0^{\pi/T} [\sigma_i^2 \psi \text{ eq } (\omega) + |\, Y \text{ eq } (\omega,\, \theta_0)\,|^2] \cdot |\, E(\omega)\,|^2\, d\omega, \tag{31}$$

subject to the same constraint equation.

The minimization problem can be solved through a straightforward application of the method of Lagrangian multipliers.

Solving

$$\frac{\partial V}{\partial E(\omega)} = \frac{\partial}{\partial E(\omega)} \left\{ \text{Re}\, \frac{T}{\pi} \int_0^{\pi/T} \{[\sigma_i^2 \psi \text{ eq } (\omega) + |\, Y \text{ eq } (\omega,\, \theta_0)\,|^2] \cdot |\, E(\omega)\,|^2 + \lambda Y \text{ eq } (\omega,\, \theta_0) E(\omega) e^{j\omega t_0}\}\, d\omega \right\} = 0, \tag{32}$$

and

$$C_1 = \text{Re}\, \frac{T}{\pi} \int_0^{\pi/T} Y \text{ eq } (\omega,\, \theta_0) E(\omega) e^{j\omega t_0}\, d\omega, \tag{33}$$

we obtain the expression for the optimum equalizer, $E_0(\omega)$, at the sampling instant, t_0, and the demodulating carrier phase, θ_0,

$$[E_0(\omega)]_{t_0,\theta_0} = \frac{\{Y\text{ eq }(\omega,\ \theta_0)e^{j\omega t_0}\}^*}{\{\sigma_i^2\psi\text{ eq }(\omega) + |\ Y\text{ eq }(\omega,\ \theta_0)\ |^2\}} \cdot \frac{C_1}{\dfrac{T}{\pi}\displaystyle\int_0^{\pi/T} \frac{|\ Y\text{ eq }(\omega,\ \theta_0)\ |^2}{\sigma_i^2\psi\text{ eq }(\omega) + |\ Y\text{ eq }(\omega,\ \theta_0)\ |^2}\, d\omega}, \tag{34}$$

where $\{X\}^*$ means complex conjugate of X. Substituting $[E_0(\omega)]_{t_0,\theta_0}$ into equation (4), we obtain the M.M.S.E.

$$[\text{M.M.S.E.}]_{t_0,\theta_0} = \frac{1}{\dfrac{T}{\pi}\displaystyle\int_0^{\pi/T} \frac{|\ Y\text{ eq }(\omega,\ \theta_0)\ |^2}{\sigma_i^2\psi\text{ eq }(\omega) + |\ Y\text{ eq }(\omega,\ \theta_0)\ |^2}\, d\omega} - 1. \tag{35}$$

Equation (35) can he further minimized over all possible sampling instants and carrier phases to obtain a global minimum of mean square error.

We now consider the class IV partial-response system. The transfer function of the equivalent baseband transmitted signal and receiver filter are

$$S(\omega_c - \omega) = R(\omega_c - \omega) = \begin{cases} \sqrt{2T \sin |\ \omega\ | T}\, e^{-j(\omega/|\omega|)\pi/4} & 0 \leqq |\ \omega\ | \leqq \dfrac{\pi}{T}, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

It follows that

$$Y\text{ eq }(\omega,\ \theta) = \begin{cases} 2T \sin \omega T e^{-j(\pi/2+\theta)\omega/|\omega|} \cdot T(\omega_c - \omega) & 0 \leqq |\ \omega\ | \leqq \dfrac{\pi}{T}, \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

The constraint equations are assumed to be

$$C = y_1(\theta_0) = \text{Re}\,\frac{1}{\pi}\int_0^{\pi/T} Y\text{ eq }(\omega,\ \theta_0)\cdot E(\omega)\cdot e^{j\omega(T+t_0)}\, d\omega, \tag{38a}$$

and

$$C - 2 = y_{-1}(\theta_0) = \text{Re}\,\frac{1}{\pi}\int_0^{\pi/T} Y\text{ eq }(\omega,\ \theta_0)\cdot E(\omega)\cdot e^{-j\omega(T-t_0)}\, d\omega, \tag{38b}$$

where C is a constant.

We now wish to minimize the function

$$U = \mathrm{Re}\,\frac{1}{\pi}\int_0^{\pi/T} \{[\sigma_i^2 2T\,|\sin\omega T| + 4T^2\,|\sin\omega T|^2\cdot|T(\omega_c-\omega)|^2]\cdot|E(\omega)|^2 + \lambda_1 Y\,\mathrm{eq}\,(\omega,\theta_0)E(\omega)e^{j\omega(T+t_0)} + \lambda_2 Y\,\mathrm{eq}\,(\omega,\theta_0)E(\omega)e^{-j\omega(T-t_0)}\}\,d\omega \tag{39}$$

subject to the constraint equations (38a) and (38b). The expression for the optimum equalizer for a given constant C, sampling instant t_0, and carrier phases θ_0 is

$$[E_0(\omega)]_{C,t_0,\theta_0} = \frac{\left[\dfrac{C(A_1-A_2)+2A_2}{A_1^2-A_2^2}\{Y\,\mathrm{eq}\,(\omega,\theta_0)e^{j\omega(T+t_0)}\}^* + \dfrac{C(A_1-A_2)-2A_1}{A_1^2-A_2^2}\{Y\,\mathrm{eq}\,(\omega,\theta_0)e^{-j\omega(T-t_0)}\}^*\right]}{\sigma_i^2\cdot 2T\,|\sin\omega T| + 4T^2\sin^2\omega T\cdot|T(\omega_c-\omega)|^2}$$

$$0 \leqq |\omega| \leqq \frac{\pi}{T}, \tag{40}$$

where

$$A_1 = \mathrm{Re}\,\frac{1}{\pi}\int_0^{\pi/T}\frac{2T\sin\omega T\cdot|T(\omega_c-\omega)|^2}{\sigma_i^2+2T\sin\omega T\cdot|T(\omega_c-\omega)|^2}\,d\omega, \tag{41a}$$

and

$$A_2 = \mathrm{Re}\,\frac{1}{\pi}\int_0^{\pi/T}\frac{2T\sin\omega T\cdot|T(\omega_c-\omega)|^2\cdot e^{j\omega 2T}}{\sigma_i^2+2T\sin\omega T\cdot|T(\omega_c-\omega)|^2}\,d\omega. \tag{41b}$$

It follows that the M.M.S.E. is

$$[\mathrm{M.M.S.E.}]_{C,t_0,\theta_0} = \frac{1}{\pi}\int_0^{\pi/T}[\sigma_i^2 2T\,|\sin\omega T| + 4T^2\,|\sin\omega T|^2\cdot|T(\omega_c-\omega)|^2] \cdot|[E_0(\omega)]_{C,t_0,\theta_0}|^2\,d\omega + 2C^2 - 4C. \tag{42}$$

Since $|Y\,\mathrm{eq}\,(\omega,\theta_0)|^2$ is independent of t_0 and θ_0, hence $|[E_0(\omega)]_{C,t_0,\theta_0}|^2$ and $[\mathrm{M.M.S.E.}]_{C,t_0,\theta_0}$ do not depend upon t_0 and θ_0.

Equation (42) can be further minimized with respect to C by solving

$$\frac{\partial}{\partial C}[\text{M.M.S.E.}]_{C,t_0,\theta_0} = 0. \tag{43}$$

The optimum solution $C = 1$, provides the M.M.S.E. for a class IV partial-response system.

$$[\text{M.M.S.E.}]_{C=1} = \frac{2}{A_1 - A_2} - 2$$

$$= \min_{\text{all } C} [\text{M.M.S.E.}]_C . \tag{44}$$

In the absence of channel noise, $\sigma_i = 0$, then

$$A_1 = 1, \tag{45}$$

and

$$A_2 = 0. \tag{46}$$

Hence,

$$[\text{M.M.S.E.}]_{C=1} = 0. \tag{47}$$

REFERENCES

1. Lucky, R. W., "Automatic Equalization for Digital Communication," B.S.T.J., *44,* No. 4 (April 1965), pp. 547–588.
2. Lucky, R. W., "Techniques for Adaptive Equalization of Digital Communication," B.S.T.J., *45,* No. 2 (February 1966), pp. 255–286.
3. Lucky, R. W., and Rudin, H. R., "An Automatic Equalizer for General Purpose Communication Channels," B.S.T.J. *46,* No. 9 (November 1967), pp. 2179–2208.
4. DiToro, M. J., "A New Method of High-Speed Adaptive Serial Communication Through Any Time-Variable and Dispersive Transmission Medium," Conf. Record, 1965 IEEE Annual Commun. Conv., pp. 763–767.
5. Coll, D. D. and George, D. A., "The Reception of Time-Dispersed Pulses," Conf. Record, 1965 IEEE Ann. Commun. Conv., pp. 749–752.
6. Chang, R. W., "Joint Optimization of Automatic Equalization and Carrier Acquisition for Digital Communication," B.S.T.J., *49,* No. 6 (July-August 1970), pp. 1069–1104.
7. Kretzmer, E. R., Generalization of a Technique for Binary Data Communication, IEEE Trans. on Commun. Tech., COM-14, No. 1 (February 1966), pp. 67–68.
8. Saltzberg, B. R., "Intersymbol Interference Error Bounds with Application to Ideal Bandlimited Signaling," IEEE Trans. Information Theory, *IT-14,* No. 4 (July 1968), pp. 563–568.